Theory of Fluid Dynamics Part 6: The maths of axisymmetric flow.

We now look at the maths involved in describing axisymmetric flow for an incompressible, irrotational (inviscid) fluid. Such a fluid is like water, in which case the topic is called Hydrodynamics. But we will pretend that air also is incompressible, in which case the topic is called Aerodynamics. The intention is to provide the equations used for the previous article, Part 5, which describe the flow of air around the nose of an axisymmetric airplane.

The type of maths concerned is called calculus, a subject which is understood by about .0001% of the Worlds population. I will make no effort to explain the calculus, unless I have a neurological fit and realise that I might understand some of it myself!

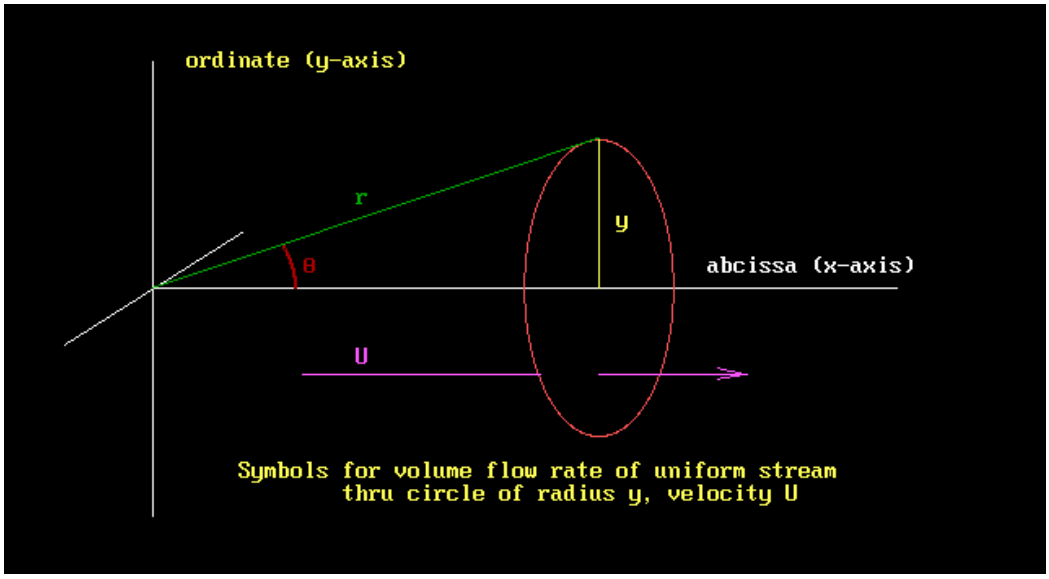
We first need an equation, which describes a uniform flow of air parallel to the abcissa (x-axis) in our rectilinear coordinate system. The stream function Ψ_1 represents the volume flow rate of such a uniform stream. It is a number, for example, we can write:

Ψ_1 = 50 litres per second for air flowing past a given area of space.

This is not very general. We really want to include some extra features, such as the velocity of the flow and the area of space of interest to our problem. Here it is:

$$\Psi_1 = -Ur^2\sin^2\theta/2$$
$$= -Uy^2/2$$

Here, U is the velocity of the uniform stream (think 'wind' or 'airspeed') along the abcissa. It has no other components, and represents a number, such as 50 metres per second. The other symbols are given in the diagram below, and "sin" is the trigonometric function, not something norty.



Now we need a "source" of fluid. In the previous work, the source was a point that injected fluid, into space, in a radial fashion, like the spokes on a bicycle wheel. This time, we will go for something more elegant. A point source injecting into an axisymmetric space will yield too blunt a representation of the aircraft spinner, although it could be good for the radial cowl of, say, a Sea Fury. We can get the both of both worlds by using the stream function for an axisymmetric line source. By making the line source short, we can get a blunt shape, and by making it long we can get a more pointy shape! If this is confusing, remember that we are going to combine the stream functions for a uniform stream and various sources, and even negative sources, which are called "sinks". The flow in sinks is reversed, flowing not out, but into the source. The mind boggles!

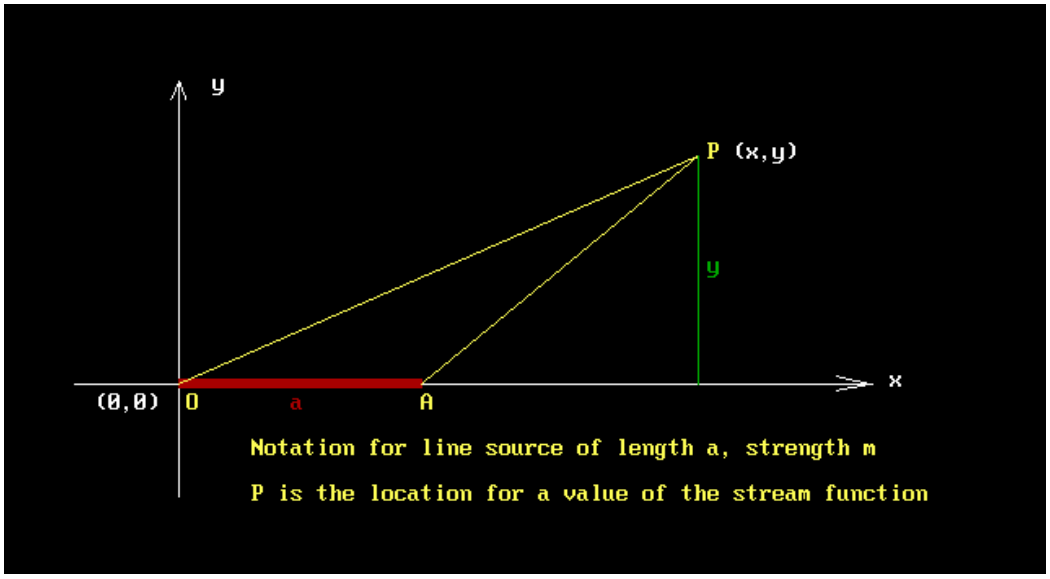
Here is the equation for the stream function of an axisymmetric line source Ψ_2

$$\Psi_2 = m(PO - PA)/a$$

It doesn't get much better than this! We have:

$$PO = (x^2 + y^2)^{1/2}$$
$$PA = [(x-a)^2 + y^2]^{1/2}$$

Here, m is the strength of the source, and a is its length. Check the diagram below.



Now we have our free stream and line source stream functions, we are allowed to add them.

$$\Psi = \Psi_1 + \Psi_2$$
$$= -Uy^2/2 + m(Po - Pa)/a$$
$$= -Uy^2/2 + m[(x^2 + y^2)^{1/2} - \{(x-a)^2 + y^2\}^{1/2}]/a$$

As this is a linear theory, we could have written in some more stream functions, say n, viz:

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3 + \ldots + \Psi n$$

Now this is really neat. With enough stream functions, we could create any shape we want! Well, maybe. Being mere mortals, we will end up using just 4 to get a good enough result for engineering purposes. Ie, within 20%. Don't let anyone tell you engineering isn't rough!

Just for the moment, we will get results for one line source only.
What we are now after are the velocity components at any point on any streamline. Lets call these u on the abcissa and v on the ordinate (x and y axes respectively). Then a result from the theory is this:

$$u = -(\partial\Psi/\partial y)/y$$

$$v = -(\partial\Psi/\partial x)/y$$

Carrying out these partial derivatives yield:

$$u = -U - m(x^2+y^2)^{-1/2}/a + m[(x-a)^2+y^2]^{-1/2}/a$$

$$v = mx(x^2+y^2)^{-1/2}/(ay) - m(x-a)[(x-a)^2+y^2]^{-1/2}/(ay)$$

Finally, it is useful to know the stagnation point. In the axisymmetric case, this occurs where u = 0 (reasonable, the stagnation point is defined as the point where there is no flow in the x-direction) and y = 0 (ie, on the axis of rotation of the axisymmetric body).

Then from our equation for u, with u = 0 and y = 0, we have:

$$U = -m(x^2)^{-1/2}/a + m[(x-a)^2]^{-1/2}/a$$
$$= m/(x(x-a))$$

If we have more than 1 line source, say n, the relations below hold:

$$u = u_1 + u_2 + \ldots + u_n$$
$$v = v_1 + v_2 + \ldots + v_n$$

I leave it to the reader to prove that these stream functions are irrotational. The required condition is given below.

$$\partial^2\Psi/\partial x^2 + \partial^2\Psi/\partial y^2 = 0$$

Finally, I append my Quick Basic code for you fellow dinosaurs out there still running DOS, or even WIN98, which look better and better to me as each day of Windows Vista passes.

To recap, we set out to get the inflow field to a propeller, as affected by the presence of the spinner and cowl. This permits a correction to the blade angle, which is determined by the local axial velocity.

Regrettably, I need to point out some apparent errors elsewhere. In “Fundamentals of Fluid Mechanics”, Editions 1 and 2, by Gerhart and Gross, equations 9.36 and 9.37 are wrong. In “Teach Yourself Calculus”, by Abbott, page 122 (section 78), the differential symbols are missing for the subjects arcsin(x/a), arctan(x/a) and arcsec(x/a).

Appendix: Quick Basic source code for axisymmetric flow.

```
     ' Source in Uniform Stream              Milne-Thompson p456
10/4/2009
     ' Axial symmetry 3-dimensional case ...                CFD by
Supercool

     ' n line sources: Milne Thompson formula
     ' Uniform stream flow over a spinner/nose/fuselage

     ' Code is Quick Basic

      DIM psi(30), x(10, 1200), y(10, 1200)      ' set up arrays
      DIM r(2000), p(2000), xx(1200), yy(1200), Uu(20), Vv(20)

      SCREEN 9, , 1, 1: pi = 4 * ATN(1): rtd = 180 / pi: dtr = pi /
180
      scl = 220          ' centimetres across screen
      WINDOW (-.1 * scl, -.3 * scl)-(.9 * scl, .3 * scl)

     ' Set input parameter values

        U = 900         ' velocity of uniform stream (cm/second)
                        ' (same thing as airspeed)
     ' spinner source parameters

        m(1) = 7000       ' source strength            (cubic
metres/stearadian)
        a(1) = 5          ' length of line source      (cm)
        offset(1) = 2     ' offset of source from origin (cm)

     ' cowl source parameters

        m(2) = 150000
        a(2) = .1
        offset(2) = 30

     ' pinch spinner/cowl source parameters

        m(3) = -90000
        a(3) = 30
        offset(3) = 30

     '  rear fuselage source parameters
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```
        m(4) = -110000
        a(4) = 200
        offset(4) = 70

     ' select sources to be represented

        startp = 1       ' Start at source startp
        stopp = 4        ' Finish at source stopp, in sequence

        xprop = 20       ' location of prop behind stagnation point
        propdia = 63.5    ' cm

    ' find values of psi (flux) for streamlines at -infinity
    ' these streamlines are for the uniform flow with zero source
strength
    ' they are chosen to give even vertical spacing of the initial
streamlines

          dely = 5                      ' spacing of initial
streamlines in cm
        FOR n = 0 TO 10
          psi(n) = -.5 * U * (n * dely) ^ 2  ' [3]'al flux for
streamline n
        NEXT n                              ' n * dely is height
above fluid axis, cm

      ' plot free-stream streamlines

        '  FOR n = 0 TO 12: LINE (-10, n * dely)-(30, n * dely), 3:
NEXT n

      ' draw line sources

              g = 0
            FOR p = startp TO stopp
              g = g + .03: COLOR 4: IF m(p) < 0 THEN COLOR 2
              LINE (offset(p), -g)-(a(p) + offset(p), g), , BF
            NEXT p

          ' stagnation point

             FOR x = -10 + .07 TO 20 STEP .1
               Uu = -U                    ' freestream velocity U
                 FOR p = startp TO stopp
                  GOSUB Uu                   ' source axial
velocities
                  Uu = Uu + uq               ' total axial velocity
                 NEXT p
               IF Uu > 0 THEN GOTO ss       ' near enough to
stagnation point
             NEXT x

ss:           LINE (x, -.3)-(x, .3), 9
              xstag = x

      ' dividing streamline occurs at psi = -Sm(p)

               psi = 0
             FOR p = startp TO stopp
               psi = psi - m(p)
             NEXT p
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```
                j = 0
            FOR x = xstag TO xstag + 260 STEP .25
              FOR y = 0 TO 25 STEP .25
                dum = psi + .5 * U * y ^ 2            ' test
inequality value
                dump = 0
               FOR p = startp TO stopp
                 GOSUB linesource
                 dump = dump + psiline        ' line source stream
function
               NEXT p
                 IF dump < dum THEN GOTO 114
             NEXT y

114             j = j + 1: xx(j) = x: yy(j) = y
                LINE -(x, y)
            NEXT x

                  jj = j
               FOR i = 2 TO jj
                IF x = xx(i - 1) THEN GOTO 115
                  LINE (xx(i - 1), yy(i - 1))-(xx(i), yy(i)), 4
                  LINE (xx(i - 1), -yy(i - 1))-(xx(i), -yy(i)), 4
               NEXT i
115
      ' fill in nose shape

            FOR k = 1 TO jj - 1
            ' IF x = xx(k - 1) THEN GOTO 117
             LINE (xx(k), yy(k))-(xx(k + 1), 0), 10, BF
             LINE (xx(k), -yy(k))-(xx(k + 1), 0), 10, BF
            NEXT k
117

       ' draw source, origin
tt:
              LINE (0, -.1)-(a(p), .1), 4, BF         ' draw source
                LINE (-.3, -.3)-(.3, .3), 12       ' mark origin, x
= 0
                LINE (-.3, .3)-(.3, -.3), 12

         ' draw prop

                xp = xstag + xprop
             LINE (xp - 2, 0)-(xp - 1, propdia / 2), 9
             LINE (xp + 2, 0)-(xp + 1, propdia / 2), 9
             LINE (xp - 1, propdia / 2)-(xp + 1, propdia / 2), 9
             LINE (xp - 2, 0)-(xp + 2, 0), 9
             PAINT (xp, propdia / 4), 2, 9

             LINE (xp - 2, 0)-(xp - 1, -propdia / 2), 9
             LINE (xp + 2, 0)-(xp + 1, -propdia / 2), 9
             LINE (xp - 1, -propdia / 2)-(xp + 1, -propdia / 2), 9
             LINE (xp - 2, 0)-(xp + 2, 0), 9
             PAINT (xp, -propdia / 4), 2, 9

       ' draw cm scale, centred at prop

           LINE (xp, -35)-(xp, 35), 12
          FOR j = -35 TO 35 STEP 5
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```
            LINE (xp, j)-(xp + 1, j), 12
          NEXT j

      ' Get axial velocities u at cm stations, location x = xprop

          x = xprop: i = 0

      FOR y = 15 TO 30 STEP 5                    ' cm stations along
prop

          i = i + 1: yy(i) = y

              dum1 = 0: dum2 = 0
         FOR p = startp TO stopp
           GOSUB velocity
           dum1 = dum1 + uq: dum2 = dum2 + vq       ' line-source
integraton for u
         NEXT p
           Uu = dum1 + U
           Vv = dum2

           Uu(i) = Uu                          ' total axial velocity
           Vv(i) = Vv                              '   "   radial   "
           Vstream(i) = SQR(Uu(i) ^ 2 + Vv(i) ^ 2)

      NEXT y


      ' draw line source

             g = 0
            FOR p = startp TO stopp
              g = g + .04: COLOR 4: IF m(p) < 0 THEN COLOR 2
             LINE (offset(p), -g)-(a(p) + offset(p), g), , BF
            NEXT p

             k = i
         FOR i = 1 TO k
           LOCATE i + 2, 36: COLOR 12
           PRINT USING "##  ####.#  ####.#  ####.#  ##.## "; yy(i);
Uu(i); Vv(i); Vstream(i); Uu(i) / U
         NEXT i
           LOCATE 1, 36: COLOR 10: PRINT "cm  Uaxial  Vradial  Wlocal
Vratio"
           COLOR 2
           LOCATE 3, 5: PRINT USING " Prop diameter   (mm)  ###.#";
propdia * 10
           LOCATE 4, 5: PRINT USING " Free stream vel (m/s) ###.#"; U
/ 10

           LOCATE 2, 5: COLOR 12: PRINT "Sea Fury"
           LOCATE 22, 8: PRINT "Axisymmetric representation of a Sea
Fury fuselage"

   ' draw desired spinner

          xs(1) = 0: ys(1) = 0
          xs(2) = 4: ys(2) = 4
          xs(3) = 10: ys(3) = 7
          xs(4) = 20: ys(4) = 10
          FOR i = 2 TO 4
```

```
          LINE (xs(i - 1) + xstag, ys(i - 1))-(xs(i) + xstag, ys(i))
          LINE (xs(i - 1) + xstag, -ys(i - 1))-(xs(i) + xstag, -
ys(i))
         NEXT i

   ' mark cowling edges as circles

             Rc = 6: Jc = xprop + 10: Kc = 35.6 / 2 - Rc: jj = -1:
GOSUB circ
             Rc = 6: Jc = xprop + 10: Kc = -35.6 / 2 + Rc: jj = 1:
GOSUB circ

     ' compute stream lines for axial symmetry

            nlines = 5
        FOR n = 0 TO nlines STEP 1                      ' each n is a
new streamline
           k = 0: stepp = .4
            FOR x = 300 TO -100 STEP -stepp
              FOR y = .01 TO 10000 STEP stepp
                psi = -.5 * U * y ^ 2            ' free stream, axial
symmetry
                 FOR p = startp TO stopp
                   GOSUB linesource
                   psi = psi + psiline            ' free stream +
sources
                 NEXT p

                IF psi > psi(n) THEN GOTO 5 ELSE GOTO 10
5             NEXT y
10              k = k + 1
                x(n, k) = x                 ' cordinates of
streamline
                y(n, k) = y
           NEXT x
                endk = k - 1              ' counter for points on
streamline
        NEXT n                            ' next streamline

     ' plot streamlines on screen

           FOR n = 1 TO nlines
             FOR k = 2 TO endk
              COLOR 14:  LINE (x(n, k - 1), y(n, (k - 1)))-(x(n, k),
y(n, k))
                LINE (x(n, k - 1), -y(n, (k - 1)))-(x(n, k), -y(n,
k))
             NEXT k
           NEXT n

          COLOR 0
          END

linesource: ' line source stream function

              '  a = length of source
              '  m = strength of source

              '  po = SQR(x ^ 2 + y ^ 2)
              '  pa = SQR((x - a) ^ 2 + y ^ 2)
              '  psi = m / a * (po - pa)
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```
              mmq = m(p) / a(p)          ' line source strength
               xq = x - offset(p)
               po = SQR(xq ^ 2 + y ^ 2)
               pa = SQR((xq - a(p)) ^ 2 + y ^ 2)
              psiline = mmq * (po - pa)

             RETURN


Uu:     ' compute axial velocity at distance x from source on axis
        ' for getting the stagnation point

              mmq = m(p) / a(p)          ' individual source
strength
               xq = x - offset(p)
               po = SQR(xq ^ 2 + y ^ 2)
               pa = SQR((xq - a(p)) ^ 2 + y ^ 2)
               xq = x - offset(p)
               uq = -mmq / xq + mmq / (xq - a(p))

         RETURN

velocity:     ' compute axial and transverse velocities
              ' at distance x,y from source

           mmq = m(p) / a(p)          ' individual source strength
            xq = x - offset(p)
            po = SQR(xq ^ 2 + y ^ 2)
            pa = SQR((xq - a(p)) ^ 2 + y ^ 2)

            dum11 = 1 / SQR(xq ^ 2 + y ^ 2)
            dum22 = 1 / SQR((xq - a(p)) ^ 2 + y ^ 2)

            uq = -mmq * dum11 + mmq * dum22
            vq = mmq * dum11 * xq / y - mmq * dum22 * (xq - a(p)) / y

         RETURN

circ:   ' True circular-arc subroutine

          ' scl = .74
          ' xmin = -7: xmax = 7
          ' ymin = xmin * scl: ymax = xmax * scl
          ' WINDOW (xmin, ymin)-(xmax, ymax)

          '    COLOR 14
          '    Rc = 5 / 2: Jc = 0: Kc = 0: jj = -1: GOSUB circ
          '    jj = 1: GOSUB circ


        IF Rc < .00001 THEN RETURN
        xoff = 100: yoff = 100               ' offsets to avoid -ve
sq. roots
        xleft = xoff - Rc: xright = xoff + Rc   '   limits x
        ylower = yoff - Rc: yupper = yoff + Rc '   limits y
        LINE (xleft - xoff + Jc, Kc)-(xleft - xoff + Jc, Kc)

          FOR ec = xleft TO xright STEP (xright - xleft) / 100
          IF ABS(ec - xoff) > ABS(Rc) THEN 2221
             yyc = SQR(Rc ^ 2 - (ec - xoff) ^ 2)
```

```
            xxc = ec - xoff
            LINE -(xxc + Jc, -yyc * jj + Kc)
2221         NEXT ec
          LINE -(xright - xoff + Jc, Kc)
          RETURN
```