Theory of Fluid Dynamics Part 7: The maths of 2 dimensional line source.

We now look at the maths involved in describing the flow for a 2-dimensional line source. As for the axisymmetric case, we consider an incompressible, irrotational (inviscid) fluid. This case of a 2-dimensional line source is of interest if we wish to determine the flow behaviour over an airfoil of infinite span. This is not as silly as it sounds, as all wing section data for lift and drag are presented for infinite aspect ratio airfoils (wings). To make these data useful for a given wing, the aspect ratio must known and plugged into the appropriate formulae.

Unfortunately, the results for the velocities are very messy, which probably explains why they are not in the text books I use! This also means I may have made a mistake somewhere in all this algebra, so readers are cordially invited to point out any errors I may have made. The casual reader may find the following of no interest, but I need the stream function and its concomitant velocities to analyse the lift of a wing section in a later article.

First we recall that the stream function Ψ for a point source is given by:

$$\Psi = -m\theta$$

Where m is the source strength and θ is the angle subtended from the source to the point of interest in the flow field. That is as clear as mud so here is a diagram.

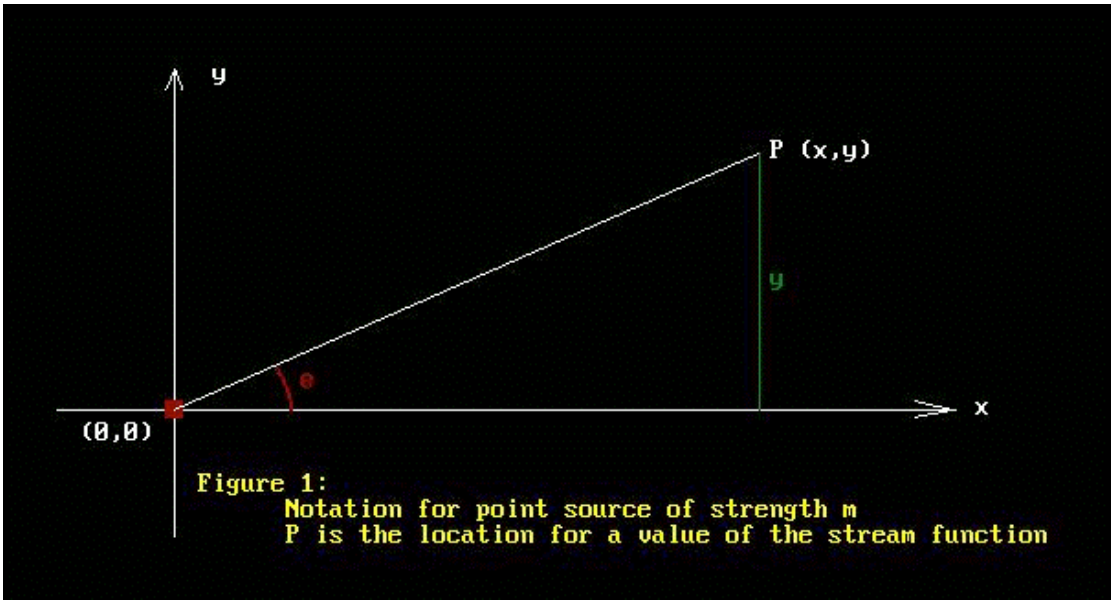

Figure 1:
Notation for point source of strength m
P is the location for a value of the stream function

The stream function for a line source can be considered to be made up of lots of small point sources, of the type above, all joined together. This time we will call the strength of the line source m_φ, where φ is the length from 0 to φ. See figure 2

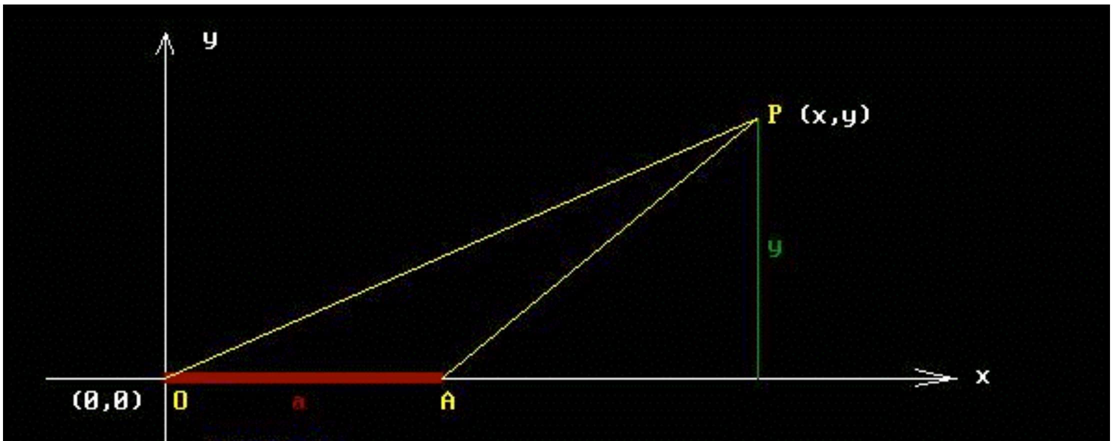

Figure 2:
Notation for line source of length a, strength m
P is the location for a value of the stream function

Using the notation in Figure 3 for the lines OP, QP and AP, we can write the stream function Ψ as:

$$\Psi = (1/a)\int_0^a m_\varphi \theta d\theta$$

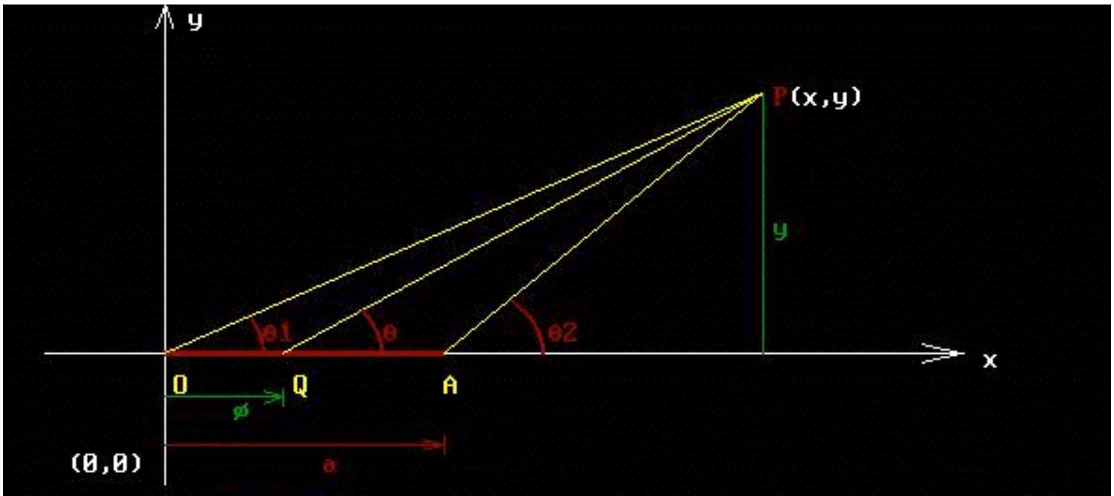

Figure 3:
Notation for [2]'al line source integration
P is the location for a value of the stream function

Noting that:

$$d(\cot\theta)/d\theta = 1/\sin^2\theta$$

and

$$\phi = x - y\cot\theta$$
$$d\phi = (y/\sin^2\theta)d\theta$$

we have:

$$\Psi = (1/a)\int_{\theta 1}^{\theta 2} (m_\varphi y\theta/\sin^2\theta).d\theta$$

Upon evaluating the integral we have:

$$\Psi = (my/a)\,[\theta 1/\tan\theta 1 - \theta 2/\tan\theta 2 + \ln\left|\sin\theta 2/\sin\theta 1\right|]$$

Which is the polar form of the [2]'al line source stream function we seek.

For computing purposes, its nice to rewrite this stream function in cartesian coordinates, when the first term in the [] braces becomes:

$$\theta 1/\tan\theta 1 - \theta 2/\tan\theta 2 = (x/y)\tan^{-1}(y/x) - ((x-a)/y)\tan^{-1}(y/(x-a))$$

and the second:

$$\ln\left|\sin\theta 2/\sin\theta 1\right| = \tfrac{1}{2}.\ln\left|(x^2 + y^2)/(y^2 + [x - a]^2)\right|$$

which just for fun is $= \ln(OP/AP)$

Now things are looking good. By selecting a value of Ψ we can draw the streamline for that constant value of Ψ. The streamline gives the direction of flow from the line source. However, it does not give the velocity of the flow along the streamline: certainly the velocity is not constant along the streamline.
So why worry? Well, those velocities tell us the pressure in the fluid: if the fluid is passing over the surface of a solid, say a wing, then that pressure gives us the force acting on the wing, also known as lift!
So now we need to find the velocities. If u is the velocity component along the x-axis (abcissa), and v, the velocity component along the y-axis (ordinate), we have from previous studies that:

$$u = -\partial\Psi/\partial y \qquad \text{and} \quad v = \partial\Psi/\partial x$$

Now this gets pretty messy. We need to write Ψ in cartesian coordinates to carry out these partial derivatives. This is what I reckon Ψ looks like.

$$\Psi = (mx/a)\tan^{-1}(y/x) - (m(x-a)/a)\tan^{-1}(y/(x-a)) + (my/(2a))\ln\left|(x^2 + y^2)/(y^2 + [x - a]^2)\right|$$

Then

$$u = -\partial\Psi/\partial y$$

$$= -mx^2/(a(x^2 + y^2)) + (m/a)(x-a)^2/[(x-a)^2 + y^2] + (m/a)[-(1/2)\ln\left|(x^2 + y^2)\right|$$

$$- y^2/(x^2 + y^2)] + (m/a)[(1/2)\ln\left|((x-a)^2 + y^2)\right| + y^2/[y^2+(x-a)^2]$$

and

$$v = \partial\Psi/\partial x$$

$$= (m/a)\{-xy/(x^2+y^2) + \tan^{-1}(y/x)\} + (m/a)[2y(x-a)/\{(x-a)^2+y^2\} - \tan^{-1}(y/(x-a))]$$

$$+ (myx)/\{a(x^2 + y^2)\} - \{my(x-a)\}/\{a[y^2 + (x-a)^2]\}$$

No doubt these expressions are reducible, but not my me. My computer won't quibble about how neat these expressions appear. Whatever, now we need some code to see if these formula give a sensible looking result.

References:

1. Tables of Integrals and other Mathematical Data. Herbert Bristol Dwight, 4th. Edition, MacMillan Publishing Co., Inc. 1961
2. Theoretical Hydrodynamics. L.M.Milne-Thompson, 4th Edition, MacMillan & Co Ltd, 1960